The Effective String

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1 Introduction

String theory enjoys applicability to a wide assortment of physical systems. For example, confining flux tubes in QCD are realized as string objects, and can be described by bosonic string theory in $D = 4$ dimensions. More generally, a long string can be seen as a collection of $D - 2$ massless Goldstone modes X^i which describe the oscillation transverse to the string (in the directions of broken spacelike translation).

Most cases of interest in effective string theory arise when the underlying theory enjoys Poincare invariance. It is more convenient to write the string embedding in Minkowski spacetime in a covariant manner $X^\mu(\sigma, \tau)$, where (σ, τ) are worldsheet parameters. We should then expect the action to be invariant under worldsheet diffeomorphisms (reparametrization) as well as spacetime Poincare symmetry. As is well known, the effective string can be described by the so-called Nambu-Goto (NG) action:

$$S_{\rm NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det h_{ab}}, \tag{1}$$

where

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu. \tag{2}$$

In the limit of a *long* string, where the finite string thickness is neglected, the NG action fully describes the string theory. More generally, the full effective-theory action is given by Poincare-invariant contributions constructed from the X^μ:

$$S_{\rm eff} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det h_{ab}}\,[1 + \cdots] \tag{3}$$

where the $\cdots$ denote higher-derivative operators constructed from worldsheet invariants, such as the second fundamental form Ω^μ_{ab}:

$$\Omega^\mu_{ab} = \nabla_a \nabla_b X^\mu. \tag{4}$$

Here the covariant derivative ∇_a is with respect to the worldsheet metric. Throughout this paper, we will consider only the long-string case, discarding such finite-size effects.

1.1 The NG Action and the Static Gauge

In formulating the effective theory of a long string, it is convenient to use diffeomorphism invariance to choose a gauge for the string embedding X^μ. A common choice for effective field theorists is the gauge $\tau = X^0$ and $\sigma = X^1$; this choice is known as the *static gauge*. This corresponds to a static string oriented spatially along the X^1-direction.

Fixing the gauge can be seen as breaking the D-dimensional spacetime Poincare invariance by choosing a long string configuration. This configuration breaks the spacetime Poincare group Poincare$(D-1, 1)$ to Poincare$(1, 1) \times SO(D-2)$; the Goldstone modes are the transverse oscillations on the string. Here, the group Poincare$(1, 1)$ corresponds to the symmetry group preserving the long string, while the $SO(D-2)$ corresponds to the residual spacetime symmetry among the Goldstones.

The "worldsheet Poincare" group can be seen to emerge if we consider the form of the worldsheet metric h_{ab} in the static gauge:

$$h_{ab} = \eta_{ab} + \partial_a X^i \partial_a X^i, \tag{5}$$

where η_{ab} is the 1+1-dimensional Minkowski metric for the worldsheet degrees of freedom. The statement of Poincare$(1,1)$-invariance of the static string configuration is then that operators in the static-string action should be invariant under transformations preserving the metric η_{ab} (of course, the full NG action should preserve the full Poincare$(D-1,1)$ symmetry). Indeed, computing $\det h_{ab}$, we see

$$\det h_{ab} = (1 - \partial_0 X^i \partial_0 X^i)(1 + \partial_1 X^i \partial_1 X^i) - (\partial_0 X^i \partial_1 X^i)(\partial_0 X^j \partial_1 X^j). \tag{6}$$

Thus,

$$\begin{aligned} S_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int dX^0 dX^1 \sqrt{(1 - \partial_0 X^i \partial_0 X^i)(1 + \partial_1 X^i \partial_1 X^i) - (\partial_0 X^i \partial_1 X^i)(\partial_0 X^j \partial_1 X^j)} \\ = -\frac{1}{2\pi\alpha'} \int \left[1 - \frac{1}{2}\left[(\partial_0 X^i)^2 - (\partial_1 X^i)^2\right] + \cdots\right] \\ = -\frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_a X^i)^2 + \cdots \equiv S^{(2)}_{\text{NG}} + S^{(4)}_{\text{NG}} + \cdots . \end{aligned} \tag{7}$$

Here the term $S^{(2)}_{\text{NG}}$ denotes the free term in the action, while $S^{(4)}_{\text{NG}}$ denotes quartic corrections to the free static-gauge action, and so on. We thus see explicitly that $S^{(2)}_{\text{NG}}$ is invariant under the broken symmetry group Poincare$(1,1) \times SO(D-2)$: Our intuition of an effective string in D-dimensions as $D-2$ transverse, massless Goldstones propagating along the string is confirmed. Interactions between these string modes are given by the quartic and higher-order corrections. The first (quartic) corrections are calculated to be

$$\begin{aligned} S^{(4)}_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int dX^0 dX^1 \Bigg[-\frac{1}{8}\left[(\partial_0 X^i \partial_0 X^i)^2 + (\partial_1 X^i \partial_1 X^i)^2\right] - \\ \frac{1}{2}\left[(\partial_0 X^i \partial_0 X^i)(\partial_1 X^i \partial_1 X^i) + (\partial_0 X^i \partial_1 X^i)(\partial_0 X^i \partial_1 X^i)\right] \Bigg] = \\ -\frac{1}{8\pi\alpha'} \int d^2\sigma \left[\frac{1}{2}(\partial_a X^i \partial^a X^i)^2 - (\partial_a X^i \partial_b X^i \partial^a X^j \partial^b X^j)\right]. \end{aligned} \tag{8}$$

Now, the terms

$$(\partial_a X^i \partial^a X^i)^2, \quad (\partial_a X^i \partial_b X^i \partial^a X^j \partial^b X^j) \tag{9}$$

are the only terms fourth-order in derivatives ∂_a and fields X^i that are invariant under the group Poincare$(1,1) \times SO(D-2)$. We can thus express the generic quartic corrections to the effective-theory static-gauge action $S^{(2)}_{\text{NG}}$ as

$$S^{(4)}_{\text{NG}} \equiv S_2 + S_3 = -\frac{1}{8\pi\alpha'} \int d^2\sigma \left[c_2(\partial_a X^i \partial^a X^i)^2 + c_3(\partial_a X^i \partial_b X^i \partial^a X^j \partial^b X^j)\right]. \tag{10}$$

For the choice $c_2 = \frac{1}{2}$, $c_3 = -1$, we reproduce the NG action to this order in the derivative expansion.

By following the standard canonical quantization procedure in $d = 2$ worldsheet dimensions, we should be able to compute S-matrix elements between worldsheet degrees of freedom in the effective theory. This should provide a perturbative realization of the effective string interactions in the long-string limit and should in principle reconcile with lattice calculations for the QCD string.

1.2 The Polyakov Action and the Composite-Liouville Term

The above discussion of the effective theory of a long string is especially important when considering fundamental strings in $D = 26$ dimensions. In the case of a fundamental string at the critical dimension, the NG action is not subject to any UV corrections, so we should expect the UV-completion of the effective theory to be the NG theory itself.

However, it is well known that for $D \neq 26$, any quantization procedure runs into problems. In light-cone quantization, as we showed on the first problem set, Lorentz-invariance is violated. In BRST quantization, unitarity is lost as negative-norm physical states are introduced.

Indeed, the problems associated with $D \neq 26$ can be traced back to the Weyl anomaly from the Polyakov path integral. Recall that the fundamental string can be equally well described by the Polyakov action

$$S_{\text{P}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X_\mu. \tag{11}$$

The Polyakov action is classically equivalent to the NG action, and it is classically invariant under both diffeomorphisms and Weyl transformations:

$$\text{Diff}: \; \delta X^\mu = \delta v^\alpha \partial_a X^\mu, \quad \delta g_{ab} = -\nabla_a \delta v_b - \nabla_b \delta v_a; \tag{12}$$

$$\text{Weyl}: \; \delta X^\mu = 0, \quad \delta g_{ab} = 2\delta\omega g_{ab} \tag{13}$$

for some infinitesimal diffeomorphism and Weyl parameters δv^a and $\delta\omega$. The Polyakov path integral is

$$\mathcal{Z}_g = \int \mathcal{D}g_{ab}\mathcal{D}X^\mu e^{iS_{\text{P}}[g_{ab},X^\mu]}. \tag{14}$$

[Strictly speaking, it is only possible to define this path integral via analytic continuation to the Euclidean signature, but for our purposes, it will suffice to write down the Minkowski path integral at a formal level.] The path integral preserves diffeomorphism invariance, but the measure $\mathcal{D}g_{ab}$ suffers an anomaly upon an infinitesimal transformation.

To see how this happens, note that we can use diffeomorphism and Weyl symmetry to fix g_{ab} to be a fiducial metric $\hat{g}_{ab}$. The choice $\hat{g}_{ab} = \eta_{ab}$ is the *conformal gauge*. [This gauge-fixing process will introduce ghosts, but we will suppress them as they are not relevant to our discussion of the Weyl anomaly.] Choosing the conformal gauge $\hat{g}_{ab} = \eta_{ab}$ and performing a Weyl transformation $g_{ab} \to e^{2\omega} g_\omega$, the path integral changes by

$$\mathcal{Z}_{e^{2\omega}\hat{g}} = \mathcal{Z}_{\hat{g}} \exp\left[-i\frac{D-26}{24\pi}\int d^2\sigma\sqrt{-g}\left[\hat{g}^{ab}\partial_a\omega\partial_b\omega - \omega R(\hat{g})\right]\right]. \tag{15}$$

Thus, the Polyakov action by itself is anomalous for $D \neq 26$. To write down a consistent theory in $D \neq 26$, the simplest and most natural way to proceed is to introduce an additional dynamical field which absorbs the Weyl anomaly introduced by the Polyakov action. To this end, we write down the *Composite-Liouville (CL) action*:

$$S_{\text{CL}} = -\frac{26-D}{24\pi}\int d^2\sigma\sqrt{-g}\left[g^{ab}\partial_a\phi\partial_b\phi - \phi R(\hat{g})\right] \tag{16}$$

where the field ϕ satisfies

$$\phi \to \phi + \omega \tag{17}$$

under a Weyl transformation. The simplest such field is given by

$$\phi = -\frac{1}{2}\log(g^{ab}\partial_a X^\mu \partial_b X_\mu). \tag{18}$$

It is now convenient to take the *conformal gauge* and choose $\hat{g}_{ab} = \eta_{ab}$. We then have $R(\hat{g}) = 0$, so we calculate

$$S_{\mathrm{CL}} = -\frac{26-D}{24\pi}\int d^2\sigma\sqrt{-g}\left[\frac{(\partial_a\partial_c X^\mu \partial^c X_\mu)(\partial^a\partial_c X^\mu \partial^c X_\mu)}{(\partial_b X^\mu \partial^b X_\mu)^2}\right] \tag{19}$$

For an anomaly-free worldsheet theory of a string in $D \neq 26$ dimensions in the conformal gauge, we can write down the following action:

$$S = S_{\mathrm{P}} + S_{\mathrm{CL}} = \int d^2\sigma\left[-\frac{1}{4\pi\alpha'}(\partial_a X^\mu)^2 - \frac{26-D}{24\pi}\left[\frac{(\partial_a\partial_c X^\mu \partial^c X_\mu)(\partial^a\partial_c X^\mu \partial^c X_\mu)}{(\partial_b X^\mu \partial^b X_\mu)^2} + \cdots\right]\right] \tag{20}$$

The CL term is highly nonlinear, and it is thus not immediately apparent if the theory is well-behaved in the UV. Nevertheless, we may still treat this as an effective theory and calculate S-matrix elements for a long string using the conformal-gauge Polyakov + CL action.

1.2.1 Asymptotic States in the Conformal Gauge and Gravitational Dressing

For the effective theory of a long string, the conformal-gauge action equally well describes the propagation of the transverse modes. However, it is important to note that the S-matrix elements from the conformal gauge are *not* the same as those in the static gauge [1]; this discrepancy is known as *gravitational dressing*. This discrepancy arises due to the fact that the asymptotic states in the conformal gauge do not correspond directly to the oscillators of the static-gauge X^i. This can be seen immediately in $D = 26$ dimensions, where conformal-gauge action is exactly free in the X^μ field, while in the static gauge, the action still has nontrivial quartic vertices.

This discrepancy, can also be seen by inspecting the quantization procedure in the conformal gauge. To begin, note that there is a large residual gauge freedom after the conformal condition $\hat{g}_{ab} = \eta_{ab}$ is imposed. Indeed, any combination of diffeomorphisms and Weyl transformation leaving g_{ab} fixed preserves the conformal gauge. Thus, we see that the conformal-gauge theory is not manifestly unitary; fixing the residual gauge freedom leads to additional ghost terms. The physical states in the conformal gauge can be constructed as the BRST cohomology of this ghost system, which will not neccessarily yield the same asymptotic states as the static-gauge X^i oscillators.

Nevertheless, the asymptotic X^i states in the two gauges should differ only by an overall conformal factor, so it is still possible to make meaningful comparisons between the conformal-gauge S-matrix elements with the corresponding static-gauge S-matrix elements. We will discuss this further when we are calculating the $2 \to 2$ scattering amplitude in both gauges.

1.3 Static-Gauge Effective String Scattering

In the previous two sections, we saw two vastly different stories about the low-energy effective theory of a long string. In the first section, we built up the long-string effective action as the

NG action from the geometric considerations of reparametrization invariance. In choosing the static gauge, we can compute string scattering amplitudes in the effective theory. In the second section, we argued for the existence of the CL term from the quantization of the Polyakov action in the conformal gauge. Although the resulting theory is not manifestly UV-complete, it can still be treated as an effective theory for a long string.

Now, the interactions from the CL term indeed appear to introduce scattering between string modes that is different from what we would expect from the static-gauge NG action. Moreover, it is not even clear at all from the NG effective theory in the static gauge that there is anything significant about the critical dimension $D = 26$! Recall the quartic NG interaction vertices:

$$S_2 + S_3 = -\frac{1}{8\pi\alpha'} \int d^2\sigma \left[c_2(\partial_a X^i \partial^a X^i)^2 + c_3(\partial_a X^i \partial_b X^i \partial^a X^j \partial^b X^j) \right]. \tag{21}$$

The choice $c_2 = \frac{1}{2}, c_3 = -1$ produces the NG action, but any values of c_2, c_3 are consistent with the symmetries of the symmetry-broken static gauge effective theory. Thus, it is *a priori* possible that we must add additional terms to the NG action to account for the interactions from the CL term in the conformal-gauge quantization of the Polyakov action.

However, as we will show, the choice $c_2 = \frac{1}{2}, c_3 = -1$ corresponding to the NG action is in fact enforced to maintain consistency with the conformal-gauge Polyakov + CL $2 \to 2$ effective string amplitude. Moreover, although this is hardly manifest at the level of the action, we will see that the CL interaction indeed appears in the NG theory at the level of the one-loop $2 \to 2$ scattering amplitude. This confirms by direct calculation the existence of the CL interaction in the static gauge. This calculation, performed by Dubosvky et. al. [2], was the first direct demonstration of this fact.

2 Tree-Level $2 \to 2$ Scattering

Throughout this section and the next, our discussion will follow [2]. To begin our analysis of the effective string, we will consider the tree-level amplitude for $2 \to 2$ effective string scattering. As we will see, this already constrains the effective theory up to the level of quartic interactions. Moreover, it will set the stage for the one-loop computations which follow.

2.1 General Properties of Effective String Amplitdes

We may consider the general form of the $2 \to 2$ effective string scattering amplitude. Label the worldsheet momenta of the incoming string modes as p_1, p_2 and the momenta of the outgoing string modes as p_3, p_4, and suppose that these modes carry "flavor" indices i, j, k, l respectively.

Now, the $SO(D-2)$ invariance of the flavor structure restricts the amplitude to have three contributing flavor structures:

$$\mathcal{M} \propto A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk}. \tag{22}$$

By crossing symmetry, we expect

$$A(s,t,u) = B(t,s,u) = C(u,t,s). \tag{23}$$

Here we have defined as usual the Mandelstam variables s, t, u:

$$s = -(p_1+p_2)^2 = -2p_1 \cdot p_2, \quad t = -(p_1-p_3)^2 = 2p_1 \cdot p_3, \quad u = -(p_1-p_4)^2 = 2p_1 \cdot p_4. \tag{24}$$

2.1.1 Absence of Annihilations in $D = 26$

Before we begin the $2 \to 2$ effective string amplitude calculation, it is important to consider a few general properties that constrain what we expect the answer to be. To begin, consider critical bosonic string theory with $D = 26$. In particular, the conformal-gauge calculation predicts a free theory in $D = 26$ dimensions. This implies that the $2 \to 2$ amplitude in the conformal gauge will involve no "annihilations": string modes cannot stop oscillating on one direction and start oscillating in another.

As we discussed previously, the asymptotic states in the conformal gauge are not exactly the same as the static-gauge X^i modes themselves. However, since they differ by a phase only, we would still expect that scattering in the static-gauge will be purely elastic without annihilations. The static gauge $2 \to 2$ amplitude in $D = 26$ should thus be given by

$$\mathcal{M} \propto \delta_{ik}\delta_{jl}\delta(p_1-p_3)\delta(p_2-p_4) + \delta_{il}\delta_{jk}\delta(p_1-p_4)\delta(p_2-p_3). \tag{25}$$

The exact factor will not be the same as in the conformal gauge; however, we should certainly not see the $SO(D-2)$ index structure $\delta_{ij}\delta_{kl}$ corresponding to the annihilation of string modes. Using the definitions of A, B, C above, we should expect

$$A = 0. \tag{26}$$

2.1.2 Flavor and Momentum Structure of Tree-Level String Amplitudes

Now, recall that we live in $d = 2$ worldsheet dimensions. As it turns out, this will impose significant kinematic constraints on the values of s, t, u. Write each of the momenta p_i in terms of their components:

$$p_i = (E_i, P_i). \tag{27}$$

We then have $E_i = |p_i|$. Moreover, momentum conservation tells us that

$$E_1 + E_2 = E_3 + E_4, \quad p_1 + p_2 = p_3 + p_4. \tag{28}$$

Suppose without loss of generality that $E_1 = +P_1$ (otherwise we may just flip the sign of every P_i without changing the argument). We then have

$$p_1 + p_3 = (P_1 + |P_3|, P_1 \pm |P_3|), \quad p_1 + p_4 = (P_1 + |P_4|, P_1 \pm |P_4|). \tag{29}$$

Suppose now that $t, u \neq 0$. This would require $P_3 = -|P_3|$ and $P_4 = -|P_4|$. But then we see

$$E_1 + P_2 = P_1 + P_2 = P_3 + P_4 = -(E_3 + E_4) = -(E_1 + E_2), \tag{30}$$

implying that

$$E_1 \pm E_2 = -E_1 - E_2, \tag{31}$$

which is clearly absurd since $E_1, E_2 > 0$. Thus, we conclude that

$$t = 0 \quad \text{or} \quad u = 0 \tag{32}$$

and since $s + t + u = 0$,

$$s = -u \quad \text{or} \quad s = -t. \tag{33}$$

2.2 Static-Gauge NG Tree-Level Amplitude Calculation

With this in mind, we now turn our attention to the $2 \to 2$ scattering at tree level using the static-gauge NG action. To begin, we recall that the quartic terms are given by

$$S_2 = -\frac{1}{8\pi\alpha'} \int d^2\sigma \; c_2(\partial_a X^i \partial^a X^i)^2, \quad S_3 = -\frac{1}{8\pi\alpha'} \int d^2\sigma \; c_3(\partial_a X^i \partial_b X^i \partial^a X^j \partial^b X^j). \tag{34}$$

To make progress, we should write down the Feynman rules corresponding to these vertices. It will prove expedient to separate the vertices by flavor structure (the s-, t-, and u-channel processes). First, note that the free part of the action is

$$S_{\text{NG}}^{\text{free}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_a X^i)^2. \tag{35}$$

The propagator is thus accompanied by an additional factor of $(2\pi\alpha')$, so when amputating external legs to obtain the Feynman amplitude, we require a factor of $\sqrt{2\pi\alpha'}^4 = (2\pi\alpha')^2$. Applying the contractions with external legs which result in the s-channel flavor structure, we obtain the following

$$c_2 \; \delta_{ij}\delta_{kl} \text{ vertex}: \quad -\frac{i}{8\pi\alpha'} \cdot (2\pi\alpha')^2 \cdot 8\delta_{ij}\delta_{kl}(p_1 \cdot p_2)(p_3 \cdot p_4) = -i\pi\alpha' s^2 \delta_{ij}\delta_{kl}. \tag{36}$$

The combinatoric factor of 8 comes from the number of ways of contracting with external legs in the s-channel. For the c_3 vertex, we notice the following:

- There are four ways in which we can contract p_1, p_3 and p_2, p_4
- There are four ways in which we can contract p_1, p_4 and p_2, p_4

Putting this together yields

$$c_3\ \delta_{ij}\delta_{kl}\ \text{vertex}:\quad -\frac{i}{8\pi\alpha'}\cdot(2\pi\alpha')^2\cdot 4\delta_{ij}\delta_{kl}\left[(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)\right]=$$

$$-\frac{i\pi\alpha'}{2}(t^2+u^2)\delta_{ij}\delta_{kl}. \tag{37}$$

Using the identity $s+t+u=0$ (which can be derived from kinematic considerations alone), we find

$$t^2+u^2=t(-s-u)+u(-s-t)=-(t+u)s-tu=s^2-tu. \tag{38}$$

The combined s-channel interaction is thus

$$\text{Combined }\delta_{ij}\delta_{kl}\ \text{vertex}:\quad -\frac{i\pi\alpha'}{2}\left[(2c_2+c_3)s^2-2c_3tu\right]\delta^{ij}\delta^{kl}. \tag{39}$$

We now recall that the s-channel amplitude A is necessarily *zero* for $D=26$. Thus, by some ruse or another, we must have

$$A=-\frac{\pi\alpha'}{2}\left[(2c_2+c_3)s^2-2c_3tu\right]=0. \tag{40}$$

Now, recall that either t or u is zero, to the final term does vanish in $d=2$ worldsheet dimensions. However, in general, the s^2 term does *not* vanish, unless we choose c_2, c_3 such that $2c_2+c_3=0$. By scaling the fields X^μ, this in fact fixes the choice

$$c_2=\frac{1}{2},\quad c_3=-1, \tag{41}$$

which is exactly the choice corresponding to the NG action. [Actually, there is a potential ambiguity with the sign of c_2, but this ambiguity is fixed by considering the forward-scattering limit.] We therefore see that the NG choice of coefficients in the EFT is required by consistency with the conformal-gauge result. Again, we emphasize that the amplitude $\mathcal{M}$ does not look the same as the conformal-gauge $2\to 2$ amplitude in $D=26$; however, it still gives rise to elastic scattering without annihilations, as it must.

3 One-Loop $2 \to 2$ Scattering

In the previous section, we saw that consistency with the conformal-gauge result in $D = 26$ for $2 \to 2$ string scattering at tree level automatically enforces the NG choice of coefficients in the static-gauge effective theory. In this section, we will take these calculations a step further. We will see that the one-loop $2 \to 2$ scattering amplitude for the NG choice of coefficients in the static gauge remains consistent with the effective Polyakov + CL theory. Just as before, we will see that at $D = 26$, the annihilation of string modes, realized by $\delta_{ij}\delta_{kl}$ vanish in both the NG and the Polyakov + CL case.

3.1 Conformal-Gauge Polyakov + CL Amplitude

Let us first consider the tree-level amplitude in the Polyakov + CL case. In the absence of the CL interaction, we should expect no annihilations for $D = 26$. This forces the amplitude to take the form

$$\mathcal{M} \propto \delta_{ik}\delta_{jl}\delta(p_1 - p_3)\delta(p_2 - p_4) + \delta_{il}\delta_{jk}\delta(p_1 - p_4)\delta(p_2 - p_3). \tag{42}$$

This is exactly what we observed in the tree-level $2 \to 2$ calculation in the static-gauge for the NG choice of coefficients.

Now, let us include the CL interaction. As we noted earlier, the CL term contains powers of ∂X in the denominator, which may *a priori* appear pathological. Indeed, the UV structure of this theory may be complicated, but we may still treat the Polyakov + CL action as an effective action for the long string in the static gauge. Thus, making the choice $X^0 = \tau$ and $X^1 = \sigma$, we can expand the CL term in the transverse modes X^i about the background long-string configuration. We see that

$$(\partial_b X^\mu \partial^b X_\mu)^2 \simeq (1 + 1 + \cdots)^2 \simeq 4 + \cdots, \tag{43}$$

where the $\cdots$ denote terms of order ∂X^i or higher. We also have

$$(\partial_a\partial_c X^\mu \partial^c X_\mu)(\partial^a\partial_c X^\mu \partial^c X_\mu) \simeq (\partial_a\partial_b X^i \partial^b X^i)(\partial^a\partial_c X^j \partial^c X^j) \tag{44}$$

Thus, to lowest order, we have

$$S_{\mathrm{P}} + S_{\mathrm{CL}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma (\partial_a X^i)^2 - \frac{26 - D}{96\pi}\int d^2\sigma (\partial_a\partial_b X^i \partial^b X^i)(\partial^a\partial_c X^j \partial^c X^j) + \cdots. \tag{45}$$

As before, the Feynman rule due to the quartic CL interaction can be split up into three different flavor structuress. Concentrating only on the $\delta_{ij}\delta_{kl}$ piece, we notice the following:

- There are four ways to contract p_1, p_2 and p_3, p_4 with the indices b, c, in which case the a indices will contract p_1 with p_3 or p_2 with p_4.
- There are four ways to contract p_1, p_2 and p_3, p_4 with the indices b, c, in which case the a indices will contract p_1 with p_4 or p_2 with p_3.

Recalling the additional factor of $(2\pi\alpha')^2$ from amputating external legs, we find the $\delta_{ij}\delta_{kl}$ vertex:

$$\text{CL } \delta_{ij}\delta_{kl} \text{ vertex}: \quad -\frac{26 - D}{96\pi} \cdot (2\pi\alpha')^2 \delta^{ij}\delta^{kl} \cdot 4\left[(p_1 \cdot p_2)^2(p_1 \cdot p_3) + (p_1 \cdot p_2)^2(p_1 \cdot p_4)\right] =$$

$$-\frac{26-D}{96\pi}\cdot(2\pi\alpha')^2\delta^{ij}\delta^{kl}\cdot\frac{1}{2}s^2(t+u)=-\frac{D-26}{96\pi}(2\pi\alpha')^2\delta^{ij}\delta^{kl}s^3. \tag{46}$$

The full amplitude is thus

$$\mathcal{M}=-\frac{D-26}{96\pi}(2\pi\alpha')^2\left[s^3\delta^{ij}\delta^{kl}+t^3\delta^{ik}\delta^{jl}+u^3\delta^{il}\delta^{jk}\right]. \tag{47}$$

Note that for $D=26$ this entire amplitude vanishes. However, if $D\neq 26$, there is a nonzero $\delta_{ij}\delta_{kl}$ term, corresponding to annihilation of string modes.

3.2 Static-Gauge NG One-Loop Amplitude Calculation

We now return our attention to the NG effective theory in the static gauge. We will compute the diagrams for $2\to 2$ scattering to one-loop order in the static-gauge theory using the quartic vertices

$$S_2=-\frac{1}{8\pi\alpha'}\int d^2\sigma\ c_2(\partial_a X^i\partial^a X^i)^2,\quad S_3=-\frac{1}{8\pi\alpha'}\int d^2\sigma\ c_3(\partial_a X^i\partial_b X^i\partial^a X^j\partial^b X^j). \tag{48}$$

In the previous section, we computed the $\delta_{ij}\delta_{kl}$ pieces of the c_2 and c_3 vertices. Using crossing symmetry, we can write down the pieces of these vertices associated with the other flavor structures. The full vertices are as follows:

$$c_2\text{ vertex}:\quad -i4\pi\alpha'\left[\delta_{ij}\delta_{kl}(p_1\cdot p_2)(p_3\cdot p_4)+\delta_{ik}\delta_{kl}(p_1\cdot p_3)(p_2\cdot p_4)+\delta_{il}\delta_{jk}(p_1\cdot p_4)(p_2\cdot p_3)\right];$$

$$c_3\text{ vertex}:\quad -i2\pi\alpha'\big[\delta_{ij}\delta_{kl}((p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3))+$$
$$\delta_{ik}\delta_{jl}((p_1\cdot p_4)(p_2\cdot p_3)+(p_1\cdot p_2)(p_3\cdot p_4))+\delta_{il}\delta_{jk}((p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_3)(p_2\cdot p_4))\big]. \tag{49}$$

The propagator can be determined from the free part of the action and is given as follows:

$$\text{Propagator}:\quad -\frac{1}{2\pi\alpha'}\frac{i\delta^{ij}}{p^2}. \tag{50}$$

The combined c_2 and c_3 vertices from the quartic NG interaction may be written in the following form

$$\text{combined } c_2 \text{ and } c_3 \text{ vertex}:\quad -i2\pi\alpha'\left[V_s\delta_{ij}\delta_{kl}+V_t\delta_{ik}\delta_{jl}+V_u\delta_{il}\delta_{jk}\right], \tag{51}$$

where

$$V_s\equiv W(p_1,p_2;p_3,p_4)=2c_2(p_1\cdot p_2)(p_3\cdot p_4)+c_3\left[(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)\right], \tag{52}$$

and

$$V_t=W(p_1,p_3;p_2,p_4),\quad V_u=W(p_1,p_4;p_2,p_3). \tag{53}$$

Notice that

$$W(a,b;c,d)=W(b,a;c,d) \tag{54}$$

and

$$W(a,b;c,d)=W(c,d;a,b). \tag{55}$$

Now consider the diagrams contributing to the one-loop $2\to 2$ scattering amplitude. There are three diagrams: the s-, t-, and u-channel processes. In each diagram, a vertex factor

$$-i2\pi\alpha' V_{ijkl}(p_1,p_2;p_3,p_4)\equiv -i2\pi\alpha'\left[V_s\delta_{ij}\delta_{kl}+V_t\delta_{ik}\delta_{jl}+V_u\delta_{il}\delta_{jk}\right] \tag{56}$$

is placed. Finally, recalling the factor $(2\pi\alpha')^2$ from the LSZ formula for the external legs, we see that the diagrams are given by the following:

$$s\text{-channel}: \quad \frac{1}{2}(2\pi\alpha')^2 \int \frac{d^dk}{(2\pi)^d} \frac{V_{ijmn}(p_1,p_2;k,p-k)V_{mnkl}(k,p-k;p_3,p_4)}{k^2(p-k)^2}; \tag{57}$$

$$t\text{-channel}: \quad \frac{1}{2}(2\pi\alpha')^2 \int \frac{d^dk}{(2\pi)^d} \frac{V_{imkn}(p_1,k;p_3,q-k)V_{mjnl}(p_2,k;p_4,q-k)}{k^2(q-k)^2}; \tag{58}$$

$$u\text{-channel}: \quad \frac{1}{2}(2\pi\alpha')^2 \int \frac{d^dk}{(2\pi)^d} \frac{V_{imln}(p_1,k;p_4,r-k)V_{mjnk}(p_2,k;p_3,r-k)}{k^2(r-k)^2}. \tag{59}$$

Here we have let

$$p = p_1 + p_2, \quad q = p_1 - p_3, \quad r = p_1 - p_4, \tag{60}$$

whence $p^2 = -s,\ q^2 = -t,\ r^2 = -u$. The factor $\frac{1}{2}$ is a symmetry factor from interchanging the two internal lines.

The full loop integral calculations involve a large number of Feynman diagrams (around a hundred). For the sake of brevity will not present this calculation here; rather, we will simply note some important features of this calculation.

Flavor Index Traces and Dependence on D

Let us consider the s-channel diagram. The flavor structure is set by the numerator:

$$V_{ijmn}(p_1,p_2;k,p-k)V_{mnkl}(k,p-k;p_3,p_4). \tag{61}$$

Expanding this out, we see that there is a term of the form

$$V_s \cdot \delta_{ij}\delta_{mn} \cdot \cdot V_s \cdot \delta_{mn}\delta_{kl} \propto (D-2)\delta_{ij}\delta_{kl}. \tag{62}$$

On the other hand, the term $V_{ijmn}(p_1,p_2;k,p-k)V_{mnkl}(k,p-k;p_3,p_4)$ also contains structures of the following form:

$$V_s \cdot \delta_{ij}\delta_{mn} \cdot V_t \cdot \delta_{mk}\delta_{nl} \propto \delta_{ij}\delta_{kl}. \tag{63}$$

Here there is no factor of $D-2$ since there is no trace over the δ_{mn}. These two contractions can be represented by the following diagrams:

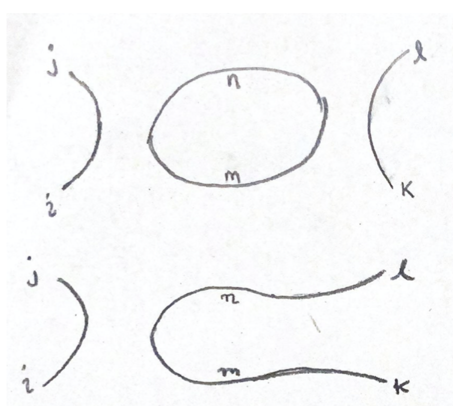

Figure 1: Two different flavor structures at one-loop contributing to an overall $\delta_{ij}\delta_{kl}$. The top diagram has an additional factor of $(D-2)$ due to the trace.

We thus see that there are diagrams proportional to $(D-2)$ as well as diagrams independent of D contributing to the overall one-loop amplitude. This opens the possibility that a specific value of D could somehow be singled out by this one-loop amplitude.

Power Counting and Dimensional Regularization

Let us first inspect the power counting of the diagrams. Notice that each vertex has four derivatives, which yield four powers of momentum. The two internal lines each have propagators, which contribute -2 powers of momentum each. The loop integration finally contributes 2 powers of momentum. The total power-counting is then

$$N = 2 \cdot 4 + 2 \cdot (-2) + 2 = 6, \tag{64}$$

so we expect the amplitude to be third power in the Mandelstam variables s, t, u. This already provides an indication that we should expect this one-loop result to correspond to the tree-level CL vertex from the conformal gauge.

Due to the above power counting, we see of course that the loop integrals are badly divergent. To regulate these divergences, we use dimensional regularization, performing the loop integrals in $d = 2 - 2\varepsilon$ dimensions. The use of dimensional regularization preserves the Poincare symmetry of the underlying NG theory; counterterms added in dimensional regularization should also be expected to preserve the NG Poincare symmetry.

Summary

From our power-counting and flavor-index analysis, we see that the one-loop amplitude should take the following schematic form:

$$\mathcal{M} = [D \cdot f_3(s,t,u) + \# \cdot g_3(s,t,u)] \cdot \delta_{ij}\delta_{kl} + \cdots . \tag{65}$$

Here $f_3(s,t,u)$ and $g_3(s,t,u)$ denote cubic functions of the Mandelstam variables. Due to the large number of diagrams to be calculated, we omit the full computation of the amplitude here (as it would take many pages). However, putting all of the integrals, we can calculate the complete contribution to the one-loop $2 \to 2$ scattering amplitude. Recall that we may write

$$i\mathcal{M}_{\text{1-loop}} = A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk}.$$

Since B, C can be obtained by crossing from A, it suffices to write down A. The UV-divergent piece of A in dimensional regularization is given by

$$A_{1/\varepsilon} = \frac{(2\pi\alpha')^2}{32\pi\varepsilon}\left[\frac{D}{2}(2c_2 + c_3)^2 s^3 - \frac{1}{3}\left(Dc_3^2 - 2c_2^2 - 22c_2c_3 - \frac{37}{2}c_3^2\right) stu\right], \tag{66}$$

and the finite (order ε^0) piece is given by

$$A_0 = -\frac{(2\pi\alpha')^2}{192\pi}\left[(D-26)s^3 + stu\left(\frac{16}{3}D + \frac{4}{3} - 2(D-8)\log\frac{-s}{\mu^2}\right) + 12tu\left(t\log\frac{s}{t} + u\log\frac{s}{u}\right)\right]. \tag{67}$$

for the NG choices $c_2 = \frac{1}{2}, c_3 = -1$. This is a remarkable result for several reasons. We will study both the UV-divergent and the finite pieces in greater detail below.

3.3 UV Divergences and Counterterms

As we see the static-gauge NG action suffers UV divergences at one-loop order. To resolve this, we should introduce a counterterm. What sort of counterterm should be introduced? To answer this question, we inspect the form of the divergent piece of the one-loop amplitude more closely. All of the terms in the amplitude have *six* powers of momentum (recall that s, t, u are quadratic in the worldsheet momenta). Thus, we should expect the counterterm to have four X^i insertions and six derivatives of the X^μ fields in the corresponding derivative expansion.

To further constrain the terms we choose, we require invariance under the flavor $SO(D-2)$. This ensures that the fields X^i should be contracted with X^i. It is then easy to see that we must choose counterterms of the form

$$(\partial^m X^i \partial^n X^i)(\partial^p X^j \partial^q X^j), \tag{68}$$

where the ∂ are chosen so that there are six derivative terms in total. Now, notice also that we must also preserve the worldsheet symmetries: The counterterms should also be invariant under the symmetry group preserving the background metric η^{ab}. This entails that derivatives should be contracted ∂_a with ∂^a, restricting the possible terms to the following form:

$$\mathcal{C}_1 = (\partial_a X^i \partial_b \partial_c X^i)(\partial^a \partial^b X^j \partial^c X^j) \;\text{ and }\; \mathcal{C}_2 = (\partial_a X^i \partial^a \partial_b X^i)(\partial_c X^i \partial^b \partial^c X^i). \tag{69}$$

Let us now consider the tree-level vertices associated with each term. Concentrating on the $\delta_{ij}\delta_{kl}$ structure, we see that the $\delta_{ij}\delta_{kl}$ piece of the term $\mathcal{C}_1$ is as follows:

$$\mathcal{C}_1 \text{ vertex}: \quad 2\cdot(2\pi\alpha')^2 i\delta^{ij}\delta^{kl}\big[(p_1\cdot p_3)(p_2\cdot p_3)(p_2\cdot p_4) + (p_2\cdot p_3)(p_1\cdot p_3)(p_1\cdot p_4) + \tag{70}$$

$$(p_1\cdot p_4)(p_2\cdot p_4)(p_2\cdot p_3) + (p_2\cdot p_4)(p_1\cdot p_4)(p_1\cdot p_3)\big] = \tag{71}$$

$$2\cdot\frac{1}{8}\cdot(2\pi\alpha')^2 2i\delta^{ij}\delta^{kl}\left[t^2u + u^2t\right] = -\frac{1}{2}(2\pi\alpha')^2 i\delta^{ij}\delta^{kl} stu. \tag{72}$$

In the first line, the factor of two up front comes from interchanging $p_1, p_2 \leftrightarrow p_3, p_4$. On the other hand, $\mathcal{C}_2$ has the following $\delta_{ij}\delta_{kl}$ contribution to its vertex:

$$\mathcal{C}_2 \text{ vertex}: \quad 2\cdot(2\pi\alpha')^2 i\delta^{ij}\delta^{kl}\big[(p_1\cdot p_2)(p_2\cdot p_4)(p_3\cdot p_4) + (p_1\cdot p_2)(p_1\cdot p_4)(p_3\cdot p_4) + \tag{73}$$

$$(p_1\cdot p_2)(p_2\cdot p_3)(p_3\cdot p_4) + (p_1\cdot p_2)(p_1\cdot p_3)(p_3\cdot p_4)\big] = \tag{74}$$

$$2\cdot\frac{1}{8}\cdot(2\pi\alpha')^2 2i\delta^{ij}\delta^{kl}\left[s^2(t+u)\right] = -\frac{1}{2}(2\pi\alpha')^2 i\delta^{ij}\delta^{kl} s^3. \tag{75}$$

As we see, the terms $\mathcal{C}_1$ and $\mathcal{C}_2$ are indeed enough to cancel the divergences in the one-loop $2 \to 2$ amplitude. We would naively expect to cancel the divergences by introducing the counterterms in dimensional regularization as follows:

$$S_{\text{counterterm}} = \frac{1}{32\pi\varepsilon}\int d^{2-2\varepsilon}\sigma\left[D(2c_2+c_3)^2\mathcal{C}_2 - \frac{2}{3}\left(Dc_3^2 - 2c_2^2 - 22c_2c_3 - \frac{37}{2}c_3^2\right)\mathcal{C}_1\right]. \tag{76}$$

There is an additional constraint. Recall that dimensional regularization preserves D-dimensional spacetime Poincare invariance, so we should expect any counterterm arising in the effective theory to be the static-gauge long string limit of some term invariant under this Poincare symmetry.

We can consider the invariant terms available to us on the worldsheet. Beyond the metric $\sqrt{-h}$ itself, the Poincare-invariant term constructed from h_{ab} at the lowest order in derivatives is given by the Einstein-Hilbert term:

$$S_{\text{EH}} = \int d^d\sigma \sqrt{-h} R. \tag{77}$$

The Einstein-Hilbert term has two derivatives of the metrix $h_{ab} = \partial_a X^\mu \partial_b X_\mu$. Thus, we should expect that the term quartic in X^i arising from $\sqrt{-h}R$ should have two more derivatives than powers of X^i, exactly as we require. Indeed, in the static gauge, we have

$$h_{ab} = \eta_{ab} + \partial_a X^i \partial_b X^i. \tag{78}$$

Treating $\partial_a X^i \partial_b X^i$ as a perturbation, we can compute the Ricci scalar to linear order in $h_{ab} = \partial_a X^i \partial_b X^i$:

$$R = \partial_a \partial_b h^{ab} - \Box h^a{}_a. \tag{79}$$

At lowest order, we may use the equations of motion $\partial^a \partial_a X^i = 0$ to set any term containing a factor of this form to zero. Integrating by parts, discarding total derivatives, and using the equations of motion we find that the Einstein-Hilbert action can be written as

$$\int d^2\sigma \sqrt{-h} R = \int d^2\sigma (\partial_a X^i \partial_b \partial_c X^i)(\partial^a \partial^b X^j \partial^c X^j) + \cdots \tag{80}$$

Thus, we see that the counterterm $\mathcal{C}_1$ is produced by including the Lorentz-invariant Einstein-Hilbert term. The counterterm $\mathcal{C}_2$, on the other hand cannot be realized by the addition of a Lorentz-invariant term to the NG action. However, this is not a problem, since for the NG choice $c_2 = \frac{1}{2},\ c_3 = -1$, the coefficient of $\mathcal{C}_2$ vanishes, and we have

$$S_{\text{counterterm}} = -\frac{D-8}{48\pi\varepsilon} \int d^{2-2\varepsilon}\sigma\ \mathcal{C}_1, \tag{81}$$

which is reproduced by introducing the counterterm

$$S_{\text{EEH}} = -\frac{D-8}{48\pi\varepsilon} \int d^{2-2\varepsilon}\sigma \sqrt{-h} R. \tag{82}$$

This term is known as the *evanescent Einstein-Hilbert term*. It is a total derivative in two dimensions, vanishing in dimensional regularization as $\varepsilon \to 0$ and $d \to 2$. Thus, it does not appear at all at tree level. However, it must be introduced in dimensional regularization to provide consistency at higher-loop order in the effective theory.

We conclude by noticing that the renormalization of the one-loop amplitude provides additional confirmation of the NG choice $c_2 = \frac{1}{2}$ and $c_3 = -1$: if $2c_2 + c_3 \neq 0$, then we would not be able to cancel the one-loop divergences by the addition of a Lorentz-invariant term to the NG action in dimensional regularization. [In fact, a term of the form $\mathcal{C}_2$ can indeed appear, but it will only appear in regularization schemes other than dimensional regularization which do not preserve Lorentz invariance.]

3.4 The Finite One-Loop Amplitude

Having studied the divergent part of the one-loop amplitude, we found that the divergence can be consistently regulated in dimensional regularization by introducing the Lorentz-invariant evanescent Einstein-Hilbert term for the NG choice of coefficients $c_2 = \frac{1}{2}$ and $c_3 = -1$. After this counterterm is added to cancel the divergence, the remaining finite part of the one-loop amplitude is given by

$$A_0 = -\frac{(2\pi\alpha')^2}{192\pi}\left[(D-26)s^3 + stu\left(\frac{16}{3}D + \frac{4}{3} - 2(D-8)\log\frac{-s}{\mu^2}\right) + 12tu\left(t\log\frac{s}{t} + u\log\frac{s}{u}\right)\right]. \tag{83}$$

Since we have subtracted off the $1/\varepsilon$ divergence with the inclusion of the counterterm, we may now safely take $\varepsilon \to 0$ and work in $d = 2$ worldsheet dimensions. In this number of dimensions, we see that $tu = 0$, so the second and third terms vanish regardless of the value of D. We therefore see that indeed $A_0 = 0$ in $D = 26$ dimensions, exactly as required.

To obtain the full amplitude, we recall that

$$\mathcal{M}_{\text{1-loop}} = A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk}. \tag{84}$$

The coefficients B and C may be obtained by crossing. Notice that the coefficient of the second summand in the corresponding amplitudes B_0, C_0 also multiplies stu; thus, this term drops out of the full amplitude in $d = 2$ worldsheet dimensions.

3.4.1 Imaginary Part: Consistency with Unitarity

Let us now look at the third summand of equation (83). In A_0, this term vanishes since $tu = 0$; however, by crossing, we obtain the term

$$12su\left(s\log\frac{t}{s} + u\log\frac{t}{u}\right)\delta ik\delta_{jl} \in B_0 \tag{85}$$

for the $\delta_{ik}\delta_{jl}$ flavor structure. Thus, for the structures B_0 and C_0, the logarithmic terms do not necessarily vanish. For instance, if we take $t = 0$ so that $s = -u$, we see that

$$B_0 = -\frac{(2\pi\alpha')^2}{192\pi}(D-26)t^3 + \frac{1}{16}(2\pi\alpha')^2 is^3 \tag{86}$$

provided that we choose the $i0$ prescription $s \to s + i0$ from the propagators.

We see that the presence of an imaginary part is consistent with unitarity. Indeed, consider the tree-level NG amplitude for $t = 0$:

$$\mathcal{M}^{ik,jl}_{\text{tree}} = -\pi\alpha' su\delta_{ik}\delta_{jl} = \frac{1}{2}(2\pi\alpha')s^2\delta_{ik}\delta_{jl}. \tag{87}$$

Unitarity demands that

$$2\text{Im}(\mathcal{M}^{ik,jl}_{\text{1-loop}}) = \frac{1}{2}\int \frac{dp_3}{(2\pi)2E_3}\frac{dp_4}{(2\pi)2E_4}\mathcal{M}^{im,jn}_{\text{tree}}(\mathcal{M}^{mk,nl}_{\text{tree}})^* \cdot (2\pi)^2\delta(p_1 + p_2 - p_3 - p_4). \tag{88}$$

The factor $\frac{1}{2}$ on the right hand side is a symmetry factor. Moving to the center-of-mass frame where $p_1 + p_2 = (E, 0)$, we see that

$$\frac{1}{2}\int \frac{dp_3}{(2\pi)2E_3}\frac{dp_4}{(2\pi)2E_4}\mathcal{M}^{im,jn}_{\text{tree}}(\mathcal{M}^{mk,nl}_{\text{tree}})^* \cdot (2\pi)^2\delta(p_1 + p_2 - p_3 - p_4) =$$

$$\frac{1}{8}(2\pi\alpha')^2 s^4\delta_{ik}\delta_{jl}\int \frac{dP_3}{2E_3}\frac{dP_4}{2E_4}\delta(P_3 + P_4)\delta(E - E_3 - E_4) =$$

$$\frac{1}{8}(2\pi\alpha')^4 s^4 \delta_{ik}\delta_{jl} \int_{-\infty}^{\infty} \frac{dP_3}{4E_3^2}\delta(E-2E_3) =$$
$$\frac{1}{8}(2\pi\alpha')^2 s^3 \delta_{ik}\delta_{jl} = 2\mathrm{Im}(B_0)\delta_{ik}\delta_{jl} = \mathrm{Im}(\mathcal{M}_{\text{1-loop}}), \quad (89)$$

exactly as required by unitarity.

Notice that this imaginary part is absent in the conformal gauge. As we discussed, the asymptotic states in the conformal gauge are different from those in the static gauge, so the S-matrix elements differ by a nontrivial factor. We have see that this accounts for the $2 \to 2$ tree-level scattering (without annihilations) in the static gauge in $D = 26$, while the conformal gauge is exactly free in this many dimensions. Thus, the imaginary part in the one-loop static-gauge amplitude is required by consistency with the static-gauge tree-level quartic vertices, but it should not appear in the corresponding conformal gauge result (where there is no nontrivial scattering in $D = 26$ dimensions).

3.4.2 Real Part: Comparison with the Polyakov + CL Amplitude

The real part of the one-loop static-gauge amplitude, accounting for all flavor structures, is then written as follows:

$$\mathcal{M}_{\mathrm{one-loop}} = -\frac{D-26}{192\pi}(2\pi\alpha')^2 \left[s^3\delta_{ij}\delta_{kl} + t^3\delta_{ik}\delta_{jl} + u^3\delta_{il}\delta_{jk}\right]. \quad (90)$$

As we now see, this exactly reproduces the CL interaction from the conformal gauge! Thus, we see that the CL interaction is generated at one-loop order in the static gauge, and it appears in the static-gauge one-loop effective action. This is a very important result: It shows that the CL interaction is non-negotiable, regardless of which gauge we work in. Although the static-gauge NG action *a priori* has no mention of the critical dimension, we see that $D = 26$ is singled out by the theory at the level of the one-loop effective action. Even from the (limited) point of view of the effective theory, the critical dimension can be seen. This also confirms to one-loop order that the static-gauge effective string scattering is elastic without annihilations in $D = 26$ dimensions to one-loop order, as it must be.

Since the asymptotic states in the static gauge differ from those in the conformal gauge, we might expect this $2 \to 2$ static-gauge S-matrix elements to deviate from the conformal-gauge result at higher-loop order by a nontrivial phase factor. However, we should still expect any term producing annihilations to be proportional to $D - 26$, as with the result we have shown here.

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