Topology and the Integer Quantum Hall Effect

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We study the Integer Quantum Hall Effect (IQHE), concentrating specifically on its connection to topology. We begin by introducing the relevant background material from the quantum theory of electrons in a magnetic field, and we then follow Laughlin [1] to fully derive the IQHE. We then discuss the topological Berry phase and study its relation to the IQHE, demonstrating that the quantization condition of the IQHE is related to the first Chern number. We also briefly discuss various applications of the topological properties of the IQHE, specifically in the study of topological insulators.

I. INTRODUCTION

The Quantum Hall Effect (QHE) is one of the richest and most interesting phenomena in condensed-matter physics. The phenomenon can be summarized as follows: What happens when you pass a perpendicular magnetic field through a material conducting a current I? [2].

The Hall Effect was first discovered by E.M. Hall in his 1879 paper *On a New Action of the Magnet on Electric Currents* [3]. In this work, Hall discovered that the application of a strong magnetic field normal to the flow of electrons led to the development of a voltage in the direction perpendicular to the current [3].

To make this picture precise, consider a two-dimensional conducting surface in the (x, y)-plane, and suppose that a current I_x passes in the $+x$-direction. Now, we turn on a magnetic field of strength B in the $+z$-direction. Hall observed that a voltage V_H develops in the y-direction proportional to the applied magnetic field [3]. This proportionality is usually expressed in terms of the transverse Hall resistivity, ρ_{xy}:

$$\rho_{xy} = \frac{B}{ne}. \tag{1}$$

Here n is the number density of conduction electrons and $-e$ is the electronic charge. The above formula holds good for small applied fields B, and is known as the *Classical Hall Effect* (CHE).

However, as the magnetic field strength is increased further, the transverse Hall voltage no longer obeys the simple linear relation shown above [2, 4]. In fact, the Hall resistivity exhibits *plateaus* at certain specific values. More precisely, over large ranges of the magnetic field, the Hall resistivity remains constant at the following values [2, 4]:

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}, \quad \nu \in \mathbb{Z}. \tag{2}$$

Here $\hbar$ is as usual Planck's constant. As we see, the resistivity assumes values proportional to the reciprocals of integers (See Figure 1). This phenomenon arises due to quantum mechanics, and is known as the *Integer Quantum Hall Effect* (IQHE).

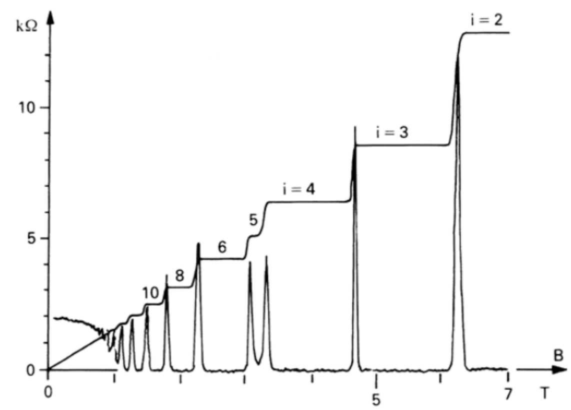

FIG. 1: This is a graph of the Hall resistivity ρ_{xy} (kΩ) as a function of the applied magnetic field B (T). Figure from [2].

More recently, it was discovered that the quantization of the Hall resistivity could be interpreted as a topological feature of the underlying quantum mechanics [2, 5, 6]. This interpretation of the IQHE is related to the geometric Berry phase and the adiabatic evolution of electron states in parameter space, and it gives a window into the deeper topological properties of quantum mechanics [2, 6, 7]. As such, the IQHE has been a source for many interesting new ideas and connections between topology and physics [2]. In fact, the idea of topological insulators was borne out of the study of the Quantum Hall Effect [8, 9]. Due to the many unique properties of topological insulators, there has also been recent interest in the application of these materials to quantum computers [2, 8, 9]. (For a further discussion of topological insulators, see [9].) Thus, the topology of the IQHE continues to be relevant to both theoretical and applied research today.

In this paper, we will discuss these topological properties of the IQHE. We will first develop the necessary background in the Classical Hall Effect and in quantum mechanics. We will then provide a complete derivation of the IQHE, and we will discuss the role of topology (specifically, Chern numbers) in the quantization of the Hall resistivity.

II. PRELIMINARY FORMALISM

In this section, we build up several of the tools from both classical and quantum physics necessary for understanding the IQHE.

A. The Classical Hall Effect

Before we discuss the Quantum Hall Effect, it is important to understand the classical phenomenon. We thus return to the picture described in the introduction: A conducting sheet in the (x, y)-plane with a current I_x in the x-direction and an applied magnetic field B in the z-direction.

Now, the general motion of a conduction electron in an electromagnetic field is given by the following [2]:

$$m_e \frac{d\boldsymbol{v}}{dt} = -e\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) - \frac{m_e \boldsymbol{v}}{\tau}. \tag{3}$$

Here, m_e is the electron mass, $\boldsymbol{v}$ is the electron velocity, $-e$ is the electronic charge, $\boldsymbol{E}$ is the electric field, and $\boldsymbol{B}$ is the magnetic field. The last term in (3) describes the retardation force experienced by conduction electrons as they scatter with the remaining atoms in the material. Here τ denotes a quantity with units of time, and can be interpreted as the time between successive scattering events.

We now restrict our attention to the case of steady-state currents. In these cases, we have

$$\frac{m_e \boldsymbol{v}}{\tau} = -e\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right). \tag{4}$$

Since the electron velocity is confined to the (x, y)-plane, we write $\boldsymbol{v}$ as a two-component vector. Then, the magnetic field $\boldsymbol{B} = (0, 0, B)$ is oriented in the z-direction, so we see that

$$\boldsymbol{v} \times \boldsymbol{B} = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix} \boldsymbol{v}. \tag{5}$$

Hence,

$$\begin{bmatrix} \frac{m_e}{\tau} & eB \\ -eB & \frac{m_e}{\tau} \end{bmatrix} \boldsymbol{v} = -e\boldsymbol{E}. \tag{6}$$

We now observe that the current density $\boldsymbol{j}$ is related to the electron velocity via $\boldsymbol{j} = -ne\boldsymbol{v}$ where n is the conduction electron number density. We find

$$\boldsymbol{E} = \frac{m_e}{ne^2\tau} \begin{bmatrix} 1 & \frac{eB\tau}{m_e} \\ -\frac{eB\tau}{m_e} & 1 \end{bmatrix} \boldsymbol{j}. \tag{7}$$

This is precisely the anisotropic form of Ohm's law:

$$\boldsymbol{E} = \begin{bmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{bmatrix} \boldsymbol{J}. \tag{8}$$

We may thus read off the transverse resistivity from (7) and (8):

$$\rho_{xy} = \frac{B}{ne}. \tag{9}$$

In the limit $t \to \infty$, the longitudinal resistivity vanishes: $\rho_{xx}, \rho_{yy} \to 0$. In this limit, the transverse conductivity σ_{xy} is easily obtained by inverting the resistivity matrix:

$$\sigma_{xy} = -\frac{1}{\rho_{xy}} = -\frac{ne}{B} \tag{10}$$

This is exactly the Classical Hall Effect as discussed in Section I. A particularly salient feature of the classical Hall resistivity is the fact that it is independent of the scattering time τ or any physics surrounding the interaction of conduction electrons with the material [2]. This indicates that the CHE is a general and exact phenomenon. We will explore this feature of the IQHE in Section III B.

B. Electrons in a Magnetic Field: Landau Levels

Having just discussed the physics behind the CHE, we will now develop some of the quantum theory underlying the IQHE. In this section, we will review the quantum dynamics of electrons in a magnetic field and derive the existence of Landau levels. To begin, we consider the Hamiltonian for an electron with charge $-e$ and mass m in an electromagnetic field:

$$H = \frac{(\boldsymbol{p} + e\boldsymbol{A})^2}{2m}. \tag{11}$$

Here $\boldsymbol{A}$ is the electromagnetic vector potential. Now, let us consider the same system analyzed for the CHE. Consider a conducting sheet in the (x, y)-plane with a magnetic field B applied in the z-direction. Since $\boldsymbol{B} = (0, 0, B)$, we may choose

$$\boldsymbol{A} = (0, Bx, 0). \tag{12}$$

This choice is known as the *Landau gauge*. Thus, the Hamiltonian is

$$H = \frac{1}{2m}\left[p_x^2 + (p_y + eBx)^2\right]. \tag{13}$$

To find the energy eigenstates $\psi_n(x, y)$, we now employ separation of variables. Write $\psi_n(x, y) = \phi_n(x)\chi_n(y)$. Since H depends on y only through p_y, $\chi_n(y)$ must be an eigenstate of p_y: $\chi_n(y) = e^{iky}$ for some k. Substituting this into the Schrodinger equation and defining

$$l_B = \sqrt{\frac{\hbar}{eB}}, \quad \omega_B = \frac{eB}{m}; \tag{14}$$

we have

$$\left[\frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_B^2(x + kl_B^2)^2\right]\phi_n(x) = E_n\phi_n(x). \tag{15}$$

This is precisely the equation for a shifted harmonic oscillator with frequency ω_B. Hence, we may immediately see

$$E_n = \hbar\omega_B\left(n+\frac{1}{2}\right). \tag{16}$$

Note also that the expectation value of x in $\phi_n(x)$ is given by the shift in the harmonic oscillator $\langle x\rangle_{\phi_n} = x^* \equiv -kl_B^2$. Thus, the general wavefunction can be written as

$$\psi_n(x,y) = e^{iky}H_n(x-x^*) \tag{17}$$

where H_n is the nth harmonic oscillator wavefunction.

The discussion thus far has considered systems of arbitrary extent in the x- and y-direction, but we now restrict the system to have finite size. We place the electron on a torus of lengths L_x and L_y in the x- and y-directions respectively. The torus forces periodic boundary conditions in the y-direction, quantizing the allowed values of k:

$$k_m = \frac{2\pi m}{L_y}, \quad m\in\mathbb{Z}. \tag{18}$$

Now, since $\langle x\rangle_{\phi_n}$ must lie within the torus, we must have $0 \le -k_m l_B^2 \le L_x$. Writing $x_m^* \equiv -k_m l_B^2$, the allowed wavefunctions are thus

$$\psi_{n,m}(x,y) = e^{ik_m y}H_n(x-x_m^*). \tag{19}$$

Counting the allowed values of m, the degeneracy N of states with energy E_n may easily be read off:

$$N = \frac{L_x L_y}{2\pi l_B^2}. \tag{20}$$

Somewhat surprisingly, the number of states in a Landau level is proportional to a macroscopic quantity, despite our discussion being entirely at the microscopic level.

The macroscopic degeneracy of Landau levels will be relevant to the IQHE: Since a large number of electrons can occupy each energy level in a magnetic field, only a "small" number of energy levels will be filled by all of the conduction electrons. The quantization condition of the IQHE will turn out to rely on the number of Landau levels filled.

III. THE INTEGER QUANTUM HALL EFFECT

Having studied the dynamics of electrons in a magnetic field, we now consider the actual IQHE. To pass from Landau quantization to the Hall conductivity requires some additional effort, and we follow an insightful derivation by Laughlin [1].

A. Derivation

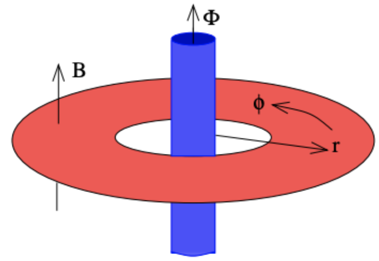

FIG. 2: A diagram of the setup described in Section III A. Note that r corresponds to our label L_x. Figure from [2].

This derivation is due to [1], and some additional explanations were found in [2]. We consider the following setup (see Fig. 2) Take a thin strip of metal of width L_x shaped into a ring in the plane of length L_y; let x be coordinate along the width of the strip, y be the tangential coordinate around the ring, and z be the perpendicular coordinate.

Suppose further that the ring is very large, so that it may essentially be approximated as a straight, flat strip at each segment. Now, suppose that a uniform radial magnetic field B passes through the strip in the z-direction. By the analysis in Section II A, the vector potential $\boldsymbol{A}_B$ in the Landau gauge corresponding to the magnetic field B is

$$\boldsymbol{A}_B = (0, Bx, 0). \tag{21}$$

Now, thread an infinitesimally thin solenoid through the center of the annulus; the solenoid carries flux Φ. We choose the vector potential $\boldsymbol{A}_\Phi$ corresponding to the threaded flux to be

$$\boldsymbol{A}_\Phi = \left(0, \frac{\Phi}{L_y}, 0\right). \tag{22}$$

Hence, just as in Section II B, the Hamiltonian is

$$H = \frac{(\boldsymbol{p}+e\boldsymbol{A})^2}{2m} = \frac{1}{2m}\left[p_x^2 + (p_y + eBx + \Phi/L_y)^2\right]. \tag{23}$$

Proceeding just as in Section II B, we write the eigenfunctions of this Hamiltonian as

$$\psi_{n,m}(x) = e^{ik_m y}H_n\left(x - x_m^* + \frac{\Phi}{BL_y}\right), \tag{24}$$

where as before we have written $x_m^* = -k_m l_B^2$ and let $H_n(x)$ be the nth harmonic oscillator wavefunction. To summarize: in the presence of a magnetic flux Φ, the electron wavefunctions on the strip are shifted in the x-direction.

Now, suppose we start with $\Phi = 0$ and slowly increase the flux Φ from 0 to Φ_0 over a long time T, where we choose the specific value

$$\Phi_0 = \frac{2\pi\hbar}{e}. \tag{25}$$

This value Φ_0 is known as the *magnetic flux quantum*. By the adiabatic theorem[1], a state initially in an energy eigenstate will remain in an energy eigenstate under a sufficiently slow variation of parameters. Hence,

$$\psi_{n,m}(x, \Phi = 0) = e^{ik_m y} H_n(x - x_m^*) \tag{26}$$

evolves to

$$\psi_{n,m}(x, \Phi = \Phi_0) = e^{ik_m y} H_n\left(x - x_m^* + \frac{2\pi\hbar}{eBL_y}\right). \tag{27}$$

But

$$x_m^* - \frac{2\pi\hbar}{eBL_y} = x_{m+1}^*, \tag{28}$$

so we in fact have the transformation

$$\psi_{n,m}(x, \Phi = 0) \to \psi_{n,m+1}(x, \Phi = \Phi_0). \tag{29}$$

Importantly, *there is no change in the overall spectrum of the theory.* This is a reflection of gauge invariance, and is related to the Aharonov-Bohm effect [1]. Now, suppose the Landau level n is fully filled. Then, all of the states $\psi_{n,m}(x)$ are occupied for

$$0 \leq x_m^* \leq L_x. \tag{30}$$

Moreover, there are several states occupying both "edges" of the strip $x = 0$ and $x = L_x$, where the potential rises steeply as the strip comes to an end. Hence, the shift $\psi_{n,m} \to \psi_{n,m+1}$ induced by the slow change $\Phi \to \Phi_0$ essentially amounts to the shift from a state on the edge $x = L_x$ to a state on the edge $x = 0$, since we see $x_{m+1}^* < x_m^*$. For every fully-filled Landau level, a single electron $-e$ is transferred from the edge $x = L_x$ to the edge $x = 0$.

We are almost done. Consider a system with n fully-filled Landau levels. As we slowly increase the threaded flux from $\Phi = 0$ to $\Phi = \Phi_0$, n electrons are transferred from $x = L_x$ to $x = 0$. Thus, there is a current density J_x in the x-direction given by

$$J_x = \frac{ne}{TL_y} \tag{31}$$

where again T is the time over which the flux changes from $\Phi = 0$ to $\Phi = \Phi_0$. At the same time, there is an electric field in the y-direction induced by the changing flux:

$$E_y = -\frac{\Phi_0}{TL_y}. \tag{32}$$

Thus, the Hall resistivity is

$$\rho_{xy} = -\frac{E_y}{J_x} = \frac{\Phi_0}{ne} = \frac{2\pi\hbar}{e^2}\frac{1}{n}, \tag{33}$$

This is exactly the result in (2), as we wished to show [1, 2].

B. Topology and the IQHE

Having derived and understood the origins of the IQHE, we now provide an alternative interpretation of the IQHE in terms Chern numbers and the Berry phase [2, 5, 10].

1. Berry Phase

This discussion is due to [2]. Before considering the IQHE in detail, we first introduce the notion of *Berry phase*. We will turn once again to the adiabatic theorem. Suppose we have a Hamiltonian $H(\theta_i)$ dependent on several parameters $\theta_i(t)$, and suppose that the $\theta_i(t)$ vary slowly in time. For an energy eigenstate $|\psi\rangle$, the Schrodinger equation yields

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H(\theta_i(t))|\psi\rangle. \tag{34}$$

Suppose that the parameters θ_i are evolved through time and brought back to their original values. By the adiabatic theorem, the resulting state $|\psi'\rangle$ must be the same as the original, which means that $|\psi'\rangle$ differs from $|\psi\rangle$ by at most a phase $e^{-i\alpha}$:

$$|\psi'\rangle = e^{-i\alpha}|\psi\rangle. \tag{35}$$

We derive a formula for this phase. Suppose that at $t = 0$,

$$|\psi(0)\rangle = |\psi_0(\theta_i(0))\rangle, \tag{36}$$

where $|\psi_0(\theta_i(t))\rangle$ is an eigenstate of the Hamiltonian at time t. A state of energy E contributes a dynamical phase $e^{-iEt/\hbar}$ upon evolution through time t regardless of how we vary the parameters; however, we are not interested in this dynamical phase, so we take the energy of $|\psi_0(\theta_i(t))\rangle$ to be zero. By the adiabatic theorem, we then see

$$H(\theta_i(t))|\psi_0(\theta_i(t))\rangle = 0 \tag{37}$$

[1] This point is a bit subtle. In fact, the adiabatic theorem does not quite hold in this case, as there are nontrivial energy level crossings. The true effect is known as *spectral flow* and is discussed in detail in [2]. However, for our purposes, it will be enough to accept the conclusion of the adiabatic theorem.

at all subsequent times t. Now, suppose that the state $|\psi(t)\rangle$ evolves as

$$|\psi(t)\rangle = U(t)|\psi_0(\theta_i(t))\rangle \tag{38}$$

for a phase $U(t)$. Differentiating with respect to time, we find that

$$|\dot{\psi}(t)\rangle = -\frac{iH}{\hbar}|\psi(t)\rangle = 0, \tag{39}$$

as we have assumed the energy of $|\psi_0(\theta(t))\rangle$ to be zero. On the other hand,

$$|\dot{\psi}(t)\rangle = \dot{U}(t)|\psi_0(\theta_i(t))\rangle + U(t)|\dot{\psi}_0(\theta_i(t))\rangle. \tag{40}$$

Dotting both sides with $\langle\psi(t)|$ and simplifying, we have

$$\dot{U}(t) = -\left\langle \psi_0(\theta_i(t)) \left| \frac{\partial}{\partial\theta_i} \right| \psi_0(\theta_i(t)) \right\rangle \frac{d\theta_i}{dt} U(t). \tag{41}$$

Then, writing

$$\mathcal{A}_i = -i\left\langle \psi_0(\theta_i(t)) \left| \frac{\partial}{\partial\theta_i} \right| \psi_0(\theta_i(t)) \right\rangle, \tag{42}$$

and solving the differential equation, we have

$$U(t) = \exp\left(-i\int_0^t dt'\ \mathcal{A}_i \frac{d\theta_i}{dt'}\right). \tag{43}$$

The quantity $\mathcal{A}_i$ is known as the *connection*, and it can be intuitively seen as the object which adiabatically "connects" states between two points in parameter space. In particular, if we evolve the parameters θ_i in time and bring them back to their initial starting point, the phase acquired is

$$U = e^{-i\alpha} \equiv \exp\left(-i\oint_C d\theta_i \mathcal{A}_i\right), \tag{44}$$

where C is the path traced in parameter space through time by the parameters θ_i. Using Stokes' theorem, we may write the contour integral as a surface integral:

$$\alpha = \int_S d\Sigma_{ij}\mathcal{F}_{ij} \tag{45}$$

where

$$\mathcal{F}_{ij} = \frac{\partial\mathcal{A}_i}{\partial\theta_j} - \frac{\partial\mathcal{A}_j}{\partial\theta_i}. \tag{46}$$

Mathematically, the quantity $\mathcal{F}_{ij}$ is the *curvature* associated with the connection $\mathcal{A}_i$: It measures how much the connection deviates from characterizing a "flat" parameter space.

2. Chern Numbers and the IQHE

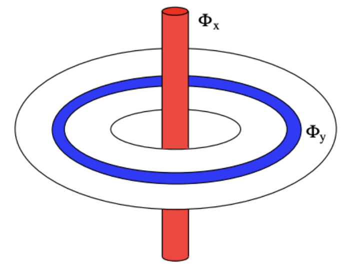

FIG. 3: A diagram of the setup described in Section III B. Figure from [2].

We now turn our attention back to the IQHE. We will show that the integer appearing in the Hall conductivity is in fact a topological invariant of a particular parameter space. Consider again two-dimensional conducting sheet of lengths L_x and L_y with a perpendicular magnetic field B [2, 5, 6]. Now, we impose periodic boundary conditions in both directions–in other words, we consider the problem on a torus. The vector potential $\boldsymbol{A}$ in the Landau gauge is

$$\boldsymbol{A} = (0, Bx, 0). \tag{47}$$

We now thread two magnetic fluxes through the torus: Φ_x and Φ_y. We let the flux Φ_x pass through the loop in the x-direction and Φ_y pass through the loop in the y-direction. (See Fig. 3.) This has the effect of shifting the vector potential to

$$\boldsymbol{A} = \left(\frac{\Phi_x}{L_x}, \frac{\Phi_y}{L_y} + Bx, 0\right). \tag{48}$$

Now, recall the magnetic flux quantum $\Phi_0 = 2\pi\hbar/e^2$. We have seen in Section III A that threading a magnetic flux Φ_0 leaves the spectrum unaltered, so we should really think of the variables Φ_x and Φ_y as periodic variables lying in the range $0 \leq \Phi_x, \Phi_y \leq \Phi_0$. Thus, we define

$$\theta_i = \frac{2\pi\Phi_i}{\Phi_0}; \tag{49}$$

then, $0 \leq \theta_i \leq 2\pi$ for $i = x, y$. We may proceed through a derivation analogous to that of Section III A to find the wavefunctions for this system:

$$\psi_{n,m}(x; \Phi_x, \Phi_y) = e^{ik_m y} \exp\left(-\frac{ie\Phi_x x}{\hbar L_x}\right) H_n\left(x - x_m^* + \frac{\Phi_y}{BL_y}\right). \tag{50}$$

Here we have again let k_m be the quantized momentum in the y-direction and $x_m^* = -k_m l_B^2$. At this point, we now compute the curvature $\mathcal{F}_{ij}$ associated with the parameters θ_x, θ_y. We have

$$\mathcal{F}_{xy} = \frac{\partial \mathcal{A}_x}{\partial \theta_y} - \frac{\partial \mathcal{A}_y}{\partial \theta_x} = -i\frac{\Phi_0^2}{(2\pi)^2}\times \left[\frac{\partial}{\partial \Phi_y}\left\langle \psi_{n,m}\middle|\frac{\partial \psi_{n,m}}{\partial \Phi_x}\right\rangle - \frac{\partial}{\partial \Phi_x}\left\langle \psi_{n,m}\middle|\frac{\partial \psi_{n,m}}{\partial \Phi_y}\right\rangle\right]. \tag{51}$$

The first term in the brackets vanishes as $\langle \psi_{n,m}|\frac{\partial \psi_{n,m}}{\partial \Phi_y}\rangle$ is independent of Φ_x. Hence, using the explicit expression for the wavefunction $\psi_{n,m}(x;\Phi_x,\Phi_y)$, we have

$$\begin{aligned}\mathcal{F}_{xy} &= -i\frac{\Phi_0^2}{(2\pi)^2}\frac{\partial}{\partial \Phi_y}\left\langle \psi_{n,m}\middle|\frac{\partial \psi_{n,m}}{\partial \Phi_x}\right\rangle \\ &= i\frac{\Phi_0^2}{(2\pi)^2}\frac{\partial}{\partial \Phi_y}\frac{ie}{\hbar L_x}\langle \psi_{n,m}|x|\psi_{n,m}\rangle \\ &= -\frac{\Phi_0^2}{(2\pi)^2}\frac{\partial}{\partial \Phi_y}\frac{e}{\hbar L_x}\left[x_m^* - \frac{\Phi_y}{BL_y}\right] \\ &= \frac{\hbar}{eBL_xL_y}\end{aligned} \tag{52}$$

Our system consists of a single electron, so the electron number density is $n = 1/(L_xL_y)$. Hence, comparing (52) with our expression for the classical Hall conductivity as a function of B in (10), we may write

$$\sigma_{xy} = -\frac{e^2}{\hbar}\mathcal{F}_{xy}. \tag{53}$$

This expression (53) turns out to hold in complete generality [2]. In fact, from this expression, we may now observe the quantization of σ_{xy}. To do this, we average $\mathcal{F}_{xy}$ over parameter space: $0 \le \theta_x, \theta_y \le 2\pi$. The conductivity should be invariant, regardless of the values of the threaded flux, so we should have

$$\sigma_{xy} = -\frac{e^2}{\hbar}\int_{[0,2\pi]^2}\frac{d^2\theta}{(2\pi)^2}\mathcal{F}_{xy}. \tag{54}$$

The quantity

$$C_1 = \int_{[0,2\pi]^2}\frac{d^2\theta}{2\pi}\mathcal{F}_{xy} \in \mathbb{Z} \tag{55}$$

is an integer known as the *(first) Chern number*, and it characterizes the topology of the parameter space integrated over. To see why $C \in \mathbb{Z}$, consider the adiabatic evolution of the parameters θ_1, θ_2 in a very, very small loop L in parameter space. Then, as the loop is contracted to a point, the Berry phase acquired should go to unity. In other words

$$e^{-i\alpha} \equiv \exp\left(-i\oint_L d\theta_i \mathcal{A}_i\right) \to 1. \tag{56}$$

Now, the whole parameter-space (θ_1, θ_2) is a surface whose boundary is the small curve L. Hence, by Stokes' theorem,

$$\oint_L d\theta_i \mathcal{A}_i = \int_{[0,2\pi]^2} d^2\theta\ \mathcal{F}_{xy}. \tag{57}$$

Thus, for the phase $e^{-i\alpha}$ to go to unity, we must have

$$\int_{[0,2\pi]^2} d^2\theta\ \mathcal{F}_{xy} \in 2\pi\mathbb{Z}, \tag{58}$$

so $C \in \mathbb{Z}$, exactly as claimed.

Putting everything together, we see

$$\sigma_{xy} = -\frac{e^2}{2\pi\hbar}C_1, \tag{59}$$

which is exactly the quantization condition of the IQHE. The quantization is seen to arise from the Chern number, a topological feature of the parameter space of the fluxes θ_x, θ_y [2].

IV. DISCUSSION

We have the Integer Quantum Hall Effect in Section III. The first argument, due to [1], uses gauge invariance. The second, due to [5], makes the connection to topology clear by relating the quantized Hall conductivity to the first Chern number of an associated parameter space. We note several salient features of the derrivations of Section III below:

1. The relationship between the IQHE and the Chern number runs much deeper than discussed in this article [2, 6, 7, 10]. The Chern number also shows up when considering the quantization of electrons on a lattice in the presence of a magnetic field, except the relevant manifold is no longer a torus in parameter space but rather the *Brillouin zone* of allowed lattice momenta. The Hall quantization is again seen to be proportional to the first Chern number, and the appearance of the Chern number in this context is known as the *TKNN invariant*. A full discussion is found in [2, 6].

2. We have explained why the IQHE quantization condition holds when the Landau levels are completely filled. From this, the experimentally-observed plateaus in the resistivity can be explained. The plateaus arise from the presence of impurities within the conductor, known as *disorder* (See [1, 2] for a full discussion.) It is notable (and somewhat ironic) that an exact, topological phenomenon is experimentally observable in part due to the "dirty" effect of impurities [2].

3. Being a topological phenomenon, the IQHE is not corrected to higher orders in perturbation theory.

Thus, the exact quantization of the Hall resistivity has been measured to remarkable accuracy [11]. It must be noted that experimental measurements of the Quantum Hall Effect have been so accurate that they have been used to *define* the value of the ratio $e^2/\hbar$ [11].

4. Though our discussion so far has been mostly theoretical, there are several real-world applications of the topological properties of the IQHE. In particular, the study of topological insulators arose primarily from the consideration of the topological properties of the IQHE [8]. As we have discussed, in the IQHE, a fixed conductivity persists over a large range of applied magnetic field strengths. A topological insulator maintains the same fixed conductivity, but does not require an applied magnetic field [8]. A unique property of these materials is that their conduction electrons do not scatter off impurities [9]. As a result, recent interest has been expressed in topological insulators due to their possible applications to quantum computing [9].

In summary, we have derived the quantization of the Hall conductivity, and we have further demonstrated that this quantization arises from a topological property (specifically, the Chern number) of the parameter space through which the electron state may adiabatically evolve. The Integer Quantum Hall Effect is a well-studied phenomenon, but a careful review of its theoretical formulation reveals many new insights about the topology underlying the physics.

Acknowledgments

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